

# Indefinite Affine Hyperspheres Admitting a Pointwise Symmetry. Part 2\*

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**Abstract.** An affine hypersurface  $M$  is said to admit a pointwise symmetry, if there exists a subgroup  $G$  of  $\text{Aut}(T_p M)$  for all  $p \in M$ , which preserves (pointwise) the affine metric  $h$ , the difference tensor  $K$  and the affine shape operator  $S$ . Here, we consider 3-dimensional indefinite affine hyperspheres, i.e.  $S = H\text{Id}$  (and thus  $S$  is trivially preserved). In Part 1 we found the possible symmetry groups  $G$  and gave for each  $G$  a canonical form of  $K$ . We started a classification by showing that hyperspheres admitting a pointwise  $\mathbb{Z}_2 \times \mathbb{Z}_2$  resp.  $\mathbb{R}$ -symmetry are well-known, they have constant sectional curvature and Pick invariant  $J < 0$  resp.  $J = 0$ . Here, we continue with affine hyperspheres admitting a pointwise  $\mathbb{Z}_3$ - or  $SO(2)$ -symmetry. They turn out to be warped products of affine spheres ( $\mathbb{Z}_3$ ) or quadrics ( $SO(2)$ ) with a curve.

*Key words:* affine hyperspheres; indefinite affine metric; pointwise symmetry; affine differential geometry; affine spheres; warped products

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## 1 Introduction

Let  $M^n$  be a connected, oriented manifold. Consider an immersed hypersurface with relative normalization, i.e., an immersion  $\varphi: M^n \rightarrow \mathbb{R}^{n+1}$  together with a transverse vector field  $\xi$  such that  $D\xi$  has its image in  $\varphi_* T_x M$ . Equi-affine geometry studies the properties of such immersions under equi-affine transformations, i.e. volume-preserving linear transformations ( $SL(n+1, \mathbb{R})$ ) and translations.

In the theory of nondegenerate equi-affine hypersurfaces there exists a canonical choice of transverse vector field  $\xi$  (unique up to sign), called the affine (Blaschke) normal, which induces a connection  $\nabla$ , a nondegenerate symmetric bilinear form  $h$  and a 1-1 tensor field  $S$  by

$$D_X Y = \nabla_X Y + h(X, Y)\xi, \quad (1)$$

$$D_X \xi = -SX, \quad (2)$$

for all  $X, Y \in \mathcal{X}(M)$ . The connection  $\nabla$  is called the induced affine connection,  $h$  is called the affine metric or Blaschke metric and  $S$  is called the affine shape operator. In general  $\nabla$  is not the Levi-Civita connection  $\hat{\nabla}$  of  $h$ . The difference tensor  $K$  is defined as

$$K(X, Y) = \nabla_X Y - \hat{\nabla}_X Y, \quad (3)$$

for all  $X, Y \in \mathcal{X}(M)$ . Moreover the form  $h(K(X, Y), Z)$  is a symmetric cubic form with the property that for any fixed  $X \in \mathcal{X}(M)$ , trace  $K_X$  vanishes. This last property is called the

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apolarity condition. The difference tensor  $K$ , together with the affine metric  $h$  and the affine shape operator  $S$  are the most fundamental algebraic invariants for a nondegenerate affine hypersurface (more details in Section 2). We say that  $M^n$  is indefinite, definite, etc. if the affine metric  $h$  is indefinite, definite, etc. (Because the affine metric is a multiple of the Euclidean second fundamental form, a positive definite hypersurface is locally strongly convex.) For details of the basic theory of nondegenerate affine hypersurfaces we refer to [7] and [10].

Here we will restrict ourselves to the case of affine hyperspheres, i.e. the shape operator will be a (constant) multiple of the identity ( $S = HId$ ). Geometrically this means that all affine normals pass through a fixed point or they are parallel. There are many affine hyperspheres, even in the two-dimensional case only partial classifications are known. This is due to the fact that affine hyperspheres reduce to the study of the Monge-Ampère equations. Our question is the following: *What can we say about a three-dimensional affine hypersphere which admits a pointwise  $G$ -symmetry, i.e. there exists a non-trivial subgroup  $G$  of the isometry group such that for every  $p \in M$  and every  $L \in G$ :*

$$K(LX_p, LY_p) = L(K(X_p, Y_p)) \quad \forall X_p, Y_p \in T_p M.$$

We have motivated this question in Part 1 [13] (see also [1, 14, 8]). A classification of 3-dimensional positive definite affine hyperspheres admitting pointwise symmetries was obtained in [14]. We continue the classification in the indefinite case. We can assume that the affine metric has index two, i.e. the corresponding isometry group is the (special) Lorentz group  $SO(1, 2)$ . In Part 1, it turns out that a  $SO(1, 2)$ -stabilizer of a nontrivial traceless cubic form is isomorphic to either  $SO(2)$ ,  $SO(1, 1)$ ,  $\mathbb{R}$ , the group  $S_3$  of order 6,  $\mathbb{Z}_2 \times \mathbb{Z}_2$ ,  $\mathbb{Z}_3$ ,  $\mathbb{Z}_2$  or it is trivial. We have shown that hyperspheres admitting a pointwise  $\mathbb{Z}_2 \times \mathbb{Z}_2$ - resp.  $\mathbb{R}$ -symmetry are well-known, they have constant sectional curvature and Pick invariant  $J < 0$  resp.  $J = 0$ .

In the following we classify the indefinite affine hyperspheres which admit a pointwise  $\mathbb{Z}_3$ -,  $SO(2)$ - or  $SO(1, 1)$ -symmetry. They turn out to be warped products of affine spheres ( $\mathbb{Z}_3$ ) or quadrics ( $SO(2)$ ,  $SO(1, 1)$ ) with a curve. As a result we get a new composition method. Since the methods for the proofs for  $SO(1, 1)$  are similar to those for  $\mathbb{Z}_3$ - or  $SO(2)$ -symmetry (and as long) we will omit them here. Both methods and results are different in case of  $S_3$ -symmetry and will be published elsewhere. The paper is organized as follows:

We will state the basic formulas of (equi-)affine hypersurface-theory needed in the further classification in Section 2. In Section 3, we show that in case of  $SO(2)$ -,  $S_3$ - or  $\mathbb{Z}_3$ -symmetry we can extend the canonical form of  $K$  (cf. [13]) locally. Thus we can obtain information about the coefficients of  $K$  and  $\nabla$  from the basic equations of Gauss, Codazzi and Ricci (cf. Section 4). In Section 5 we show, that in case of  $\mathbb{Z}_3$ - or  $SO(2)$ -symmetry it follows that the hypersurface admits a warped product structure  $\mathbb{R} \times_{ef} N^2$ . Then we classify such hyperspheres by showing how they can be constructed starting from 2-dimensional positive definite affine spheres resp. quadrics (cf. Theorems 1–8). We end in Section 6 by stating the classification results in case of  $SO(1, 1)$ -symmetry (cf. Theorems 9–16).

The classification can be seen as a generalization of the well known Calabi product of hyperbolic affine spheres [2, 6] and of the constructions for affine spheres considered in [5]. The following natural question for a (de)composition theorem, related to these constructions, gives another motivation for studying 3-dimensional hypersurfaces admitting a pointwise symmetry:

**(De)composition Problem.** *Let  $M^n$  be a nondegenerate affine hypersurface in  $\mathbb{R}^{n+1}$ . Under what conditions do there exist affine hyperspheres  $M_1^r$  in  $\mathbb{R}^{r+1}$  and  $M_2^s$  in  $\mathbb{R}^{s+1}$ , with  $r+s = n-1$ , such that  $M = I \times_{f_1} M_1 \times_{f_2} M_2$ , where  $I \subset \mathbb{R}$  and  $f_1$  and  $f_2$  depend only on  $I$  (i.e.  $M$  admits a warped product structure)? How can the original immersion be recovered starting from the immersion of the affine spheres?*

Of course the first dimension in which the above problem can be considered is three and our results provide an answer in that case.

## 2 Basics of affine hypersphere theory

First we recall the definition of the affine normal  $\xi$  (cf. [10]). In equi-affine hypersurface theory on the ambient space  $\mathbb{R}^{n+1}$  a fixed volume form  $\det$  is given. A transverse vector field  $\xi$  induces a volume form  $\theta$  on  $M$  by  $\theta(X_1, \dots, X_n) = \det(\varphi_* X_1, \dots, \varphi_* X_n, \xi)$ . Also the affine metric  $h$  defines a volume form  $\omega_h$  on  $M$ , namely  $\omega_h = |\det h|^{1/2}$ . Now the affine normal  $\xi$  is uniquely determined (up to sign) by the conditions that  $D\xi$  is everywhere tangential (which is equivalent to  $\nabla\theta = 0$ ) and that

$$\theta = \omega_h. \quad (4)$$

Since we only consider 3-dimensional indefinite hyperspheres, i.e.

$$S = HId, \quad H = \text{const}, \quad (5)$$

we can fix the orientation of the affine normal  $\xi$  such that the affine metric has signature one. Then the sign of  $H$  in the definition of an affine hypersphere is an invariant.

Next we state some of the fundamental equations, which a nondegenerate hypersurface has to satisfy, see also [10] or [7]. These equations relate  $S$  and  $K$  with amongst others the curvature tensor  $R$  of the induced connection  $\nabla$  and the curvature tensor  $\hat{R}$  of the Levi-Civita connection  $\hat{\nabla}$  of the affine metric  $h$ . There are the Gauss equation for  $\nabla$ , which states that:

$$R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY,$$

and the Codazzi equation

$$(\nabla_X S)Y = (\nabla_Y S)X.$$

Also we have the total symmetry of the affine cubic form

$$C(X, Y, Z) = (\nabla_X h)(Y, Z) = -2h(K(X, Y), Z). \quad (6)$$

The fundamental existence and uniqueness theorem, see [3] or [4], states that given  $h$ ,  $\nabla$  and  $S$  such that the difference tensor is symmetric and traceless with respect to  $h$ , on a simply connected manifold  $M$  an affine immersion of  $M$  exists if and only if the above Gauss equation and Codazzi equation are satisfied.

From the Gauss equation and Codazzi equation above the Codazzi equation for  $K$  and the Gauss equation for  $\hat{\nabla}$  follow:

$$(\hat{\nabla}_X K)(Y, Z) - (\hat{\nabla}_Y K)(X, Z) = \frac{1}{2}(h(Y, Z)SX - h(X, Z)SY - h(SY, Z)X + h(SX, Z)Y),$$

and

$$\hat{R}(X, Y)Z = \frac{1}{2}(h(Y, Z)SX - h(X, Z)SY + h(SY, Z)X - h(SX, Z)Y) - [K_X, K_Y]Z$$

If we define the Ricci tensor of the Levi-Civita connection  $\hat{\nabla}$  by:

$$\widehat{\text{Ric}}(X, Y) = \text{trace}\{Z \mapsto \hat{R}(Z, X)Y\}. \quad (7)$$

and the Pick invariant by:

$$J = \frac{1}{n(n-1)}h(K, K), \quad (8)$$

then from the Gauss equation we get for the scalar curvature  $\hat{\kappa} = \frac{1}{n(n-1)}(\sum_{i,j} h^{ij} \widehat{\text{Ric}}_{ij})$ :

$$\hat{\kappa} = H + J. \quad (9)$$

For an affine hypersphere the Gauss and Codazzi equations have the form:

$$R(X, Y)Z = H(h(Y, Z)X - h(X, Z)Y), \quad (10)$$

$$(\nabla_X H)Y = (\nabla_Y H)X, \quad \text{i.e. } H = \text{const}, \quad (11)$$

$$(\widehat{\nabla}_X K)(Y, Z) = (\widehat{\nabla}_Y K)(X, Z), \quad (12)$$

$$\hat{R}(X, Y)Z = H(h(Y, Z)X - h(X, Z)Y) - [K_X, K_Y]Z. \quad (13)$$

Since  $H$  is constant, we can rescale  $\varphi$  such that  $H \in \{-1, 0, 1\}$ .

### 3 A local frame for pointwise $\text{SO}(2)$ -, $S_3$ - or $\mathbb{Z}_3$ -symmetry

Let  $M^3$  be a hypersphere admitting a  $\text{SO}(2)$ -,  $S_3$ - or  $\mathbb{Z}_3$ -symmetry. According to [13, Theorem 4, 2.-4.] there exists for every  $p \in M^3$  an ONB  $\{\mathbf{t}, \mathbf{v}, \mathbf{w}\}$  of  $T_p M^3$  such that

$$\begin{aligned} K(\mathbf{t}, \mathbf{t}) &= -2a_4\mathbf{t}, & K(\mathbf{t}, \mathbf{v}) &= a_4\mathbf{v}, & K(\mathbf{t}, \mathbf{w}) &= a_4\mathbf{w}, \\ K(\mathbf{v}, \mathbf{v}) &= -a_4\mathbf{t} + a_6\mathbf{v}, & K(\mathbf{v}, \mathbf{w}) &= -a_6\mathbf{w}, & K(\mathbf{w}, \mathbf{w}) &= -a_4\mathbf{t} - a_6\mathbf{v}, \end{aligned}$$

where  $a_4 > 0$  and  $a_6 = 0$  in case of  $\text{SO}(2)$ -symmetry,  $a_4 = 0$  and  $a_6 > 0$  for  $S_3$ , and  $a_4 > 0$  and  $a_6 > 0$  for  $\mathbb{Z}_3$ .

We would like to extend the ONB locally. It is well known that  $\widehat{\text{Ric}}$  (cf. (7)) is a symmetric operator and we compute (some of the computations in this section are done with the CAS Mathematica<sup>1</sup>):

**Lemma 1.** *Let  $p \in M^3$  and  $\{\mathbf{t}, \mathbf{v}, \mathbf{w}\}$  the basis constructed earlier. Then*

$$\begin{aligned} \widehat{\text{Ric}}(\mathbf{t}, \mathbf{t}) &= -2(H - 3a_4^2), & \widehat{\text{Ric}}(\mathbf{t}, \mathbf{v}) &= 0, \\ \widehat{\text{Ric}}(\mathbf{t}, \mathbf{w}) &= 0, & \widehat{\text{Ric}}(\mathbf{v}, \mathbf{v}) &= 2(H - a_4^2 + a_6^2), \\ \widehat{\text{Ric}}(\mathbf{v}, \mathbf{w}) &= 0, & \widehat{\text{Ric}}(\mathbf{w}, \mathbf{w}) &= 2(H - a_4^2 + a_6^2). \end{aligned}$$

**Proof.** The proof is a straight-forward computation using the Gauss equation (13). It follows e.g. that

$$\begin{aligned} \hat{R}(\mathbf{t}, \mathbf{v})\mathbf{t} &= H\mathbf{v} - K_{\mathbf{t}}(a_4\mathbf{v}) + K_{\mathbf{v}}(-2a_4\mathbf{t}) = H\mathbf{v} - a_4^2\mathbf{v} - 2a_4^2\mathbf{v} = (H - 3a_4^2)\mathbf{v}, \\ \hat{R}(\mathbf{t}, \mathbf{w})\mathbf{t} &= H\mathbf{w} - K_{\mathbf{t}}(a_4\mathbf{w}) + K_{\mathbf{w}}(-2a_4\mathbf{t}) = H\mathbf{w} - a_4^2\mathbf{w} - 2a_4^2\mathbf{w} = (H - 3a_4^2)\mathbf{w}, \\ \hat{R}(\mathbf{t}, \mathbf{v})\mathbf{w} &= -K_{\mathbf{t}}(-a_6\mathbf{w}) + K_{\mathbf{v}}(a_4\mathbf{w}) = 0. \end{aligned}$$

From this it immediately follows that

$$\widehat{\text{Ric}}(\mathbf{t}, \mathbf{t}) = -2(H - 3a_4^2)$$

and

$$\widehat{\text{Ric}}(\mathbf{t}, \mathbf{w}) = 0.$$

The other equations follow by similar computations. ■

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<sup>1</sup>See Appendix or [http://www.math.tu-berlin.de/~schar/IndefSym\\_typ234.html](http://www.math.tu-berlin.de/~schar/IndefSym_typ234.html).

We want to show that the basis we have constructed, at each point  $p$ , can be extended differentiably to a neighborhood of the point  $p$  such that, at every point,  $K$  with respect to the frame  $\{T, V, W\}$  has the previously described form.

**Lemma 2.** *Let  $M^3$  be an affine hypersphere in  $\mathbb{R}^4$  which admits a pointwise  $\text{SO}(2)$ -,  $S_3$ - or  $\mathbb{Z}_3$ -symmetry. Let  $p \in M$ . Then there exists an orthonormal frame  $\{T, V, W\}$  defined in a neighborhood of the point  $p$  such that  $K$  is given by:*

$$\begin{aligned} K(T, T) &= -2a_4T, & K(T, V) &= a_4V, & K(T, W) &= a_4W, \\ K(V, V) &= -a_4T + a_6V, & K(V, W) &= -a_6W, & K(W, W) &= -a_4T - a_6V, \end{aligned}$$

where  $a_4 > 0$  and  $a_6 = 0$  in case of  $\text{SO}(2)$ -symmetry,  $a_4 = 0$  and  $a_6 > 0$  in case of  $S_3$ -symmetry, and  $a_4 > 0$  and  $a_6 > 0$  in case of  $\mathbb{Z}_3$ -symmetry.

**Proof.** First we want to show that at every point the vector  $\mathbf{t}$  is uniquely defined (up to sign) and differentiable. We introduce a symmetric operator  $\hat{A}$  by:

$$\widehat{\text{Ric}}(Y, Z) = h(\hat{A}Y, Z).$$

Clearly  $\hat{A}$  is a differentiable operator on  $M$ . Since  $2(H - 3a_4^2) \neq 2(H - a_4^2 + a_6^2)$ , the operator has two distinct eigenvalues. A standard result then implies that the eigen distributions are differentiable. We take  $T$  a local unit vector field spanning the 1-dimensional eigen distribution, and local orthonormal vector fields  $\tilde{V}$  and  $\tilde{W}$  spanning the second eigen distribution. If  $a_6 = 0$ , we can take  $V = \tilde{V}$  and  $W = \tilde{W}$ .

As  $T$  is (up to sign) uniquely determined, for  $a_6 \neq 0$  there exist differentiable functions  $a_4, c_6$  and  $c_7, c_6^2 + c_7^2 \neq 0$ , such that

$$\begin{aligned} K(T, T) &= -2a_4T, & K(\tilde{V}, \tilde{V}) &= -a_4T + c_6\tilde{V} + c_7\tilde{W}, \\ K(T, \tilde{V}) &= a_4\tilde{V}, & K(\tilde{V}, \tilde{W}) &= c_7\tilde{V} - c_6\tilde{W}, \\ K(T, \tilde{W}) &= a_4\tilde{W}, & K(\tilde{W}, \tilde{W}) &= -a_4T - c_6\tilde{V} - c_7\tilde{W}. \end{aligned}$$

As we have shown in [13], in the proof of Theorem 2 (Case 2), we can always rotate  $\tilde{V}$  and  $\tilde{W}$  such that we obtain the desired frame. ■

**Remark 1.** It actually follows from the proof of the previous lemma that the vector field  $T$  is (up to sign) invariantly defined on  $M$ , and therefore the function  $a_4$ , too. Since the Pick invariant (8)  $J = \frac{1}{3}(-5a_4^2 + 2a_6^2)$ , the function  $a_6$  also is invariantly defined on the affine hypersphere  $M^3$ .

## 4 Gauss and Codazzi for pointwise $\text{SO}(2)$ -, $S_3$ - or $\mathbb{Z}_3$ -symmetry

In this section we always will work with the local frame constructed in the previous lemma. We denote the coefficients of the Levi-Civita connection with respect to this frame by:

$$\begin{aligned} \hat{\nabla}_T T &= a_{12}V + a_{13}W, & \hat{\nabla}_T V &= a_{12}T - b_{13}W, & \hat{\nabla}_T W &= a_{13}T + b_{13}V, \\ \hat{\nabla}_V T &= a_{22}V + a_{23}W, & \hat{\nabla}_V V &= a_{22}T - b_{23}W, & \hat{\nabla}_V W &= a_{23}T + b_{23}V, \\ \hat{\nabla}_W T &= a_{32}V + a_{33}W, & \hat{\nabla}_W V &= a_{32}T - b_{33}W, & \hat{\nabla}_W W &= a_{33}T + b_{33}V. \end{aligned}$$

We will evaluate first the Codazzi and then the Gauss equations ((12) and (13)) to obtain more information.

**Lemma 3.** *Let  $M^3$  be an affine hypersphere in  $\mathbb{R}^4$  which admits a pointwise  $\text{SO}(2)$ -,  $S_3$ - or  $\mathbb{Z}_3$ -symmetry and  $\{T, V, W\}$  the corresponding ONB. If the symmetry group is*

**SO(2)**, then  $0 = a_{12} = a_{13} = a_{23} = a_{32}$ ,  $a_{33} = a_{22}$  and

$$T(a_4) = -4a_{22}a_4, \quad 0 = V(a_4) = W(a_4),$$

**S<sub>3</sub>**, then  $0 = a_{12} = a_{13}$ ,  $a_{23} = -3b_{13} = -a_{32}$ ,  $a_{33} = a_{22}$  and

$$T(a_6) = -a_{22}a_6, \quad V(a_6) = 3b_{33}a_6, \quad W(a_6) = -3b_{23}a_6,$$

**$\mathbb{Z}_3$  and  $\mathbf{a}_6^2 \neq 4\mathbf{a}_4^2$** , then  $0 = a_{12} = a_{13} = a_{23} = a_{32}$ ,  $a_{33} = a_{22}$ ,  $b_{13} = 0$ ,

$$T(a_4) = -4a_{22}a_4, \quad 0 = V(a_4) = W(a_4), \quad \text{and}$$

$$T(a_6) = -a_{22}a_6, \quad V(a_6) = 3b_{33}a_6, \quad W(a_6) = -3b_{23}a_6,$$

**$\mathbb{Z}_3$  and  $\mathbf{a}_6 = 2\mathbf{a}_4$** , then  $a_{12} = 2a_{22} = -2a_{33} = -b_{33}$ ,

$$a_{13} = -2a_{23} = -2a_{32} = b_{23}, \quad b_{13} = 0, \quad \text{and}$$

$$T(a_4) = 0, \quad V(a_4) = -4a_{22}a_4, \quad W(a_4) = 4a_{23}a_4.$$

**Proof.** An evaluation of the Codazzi equations (12) with the help of the CAS Mathematica leads to the following equations (they relate to eq1–eq6 and eq8–eq9 in the Mathematica notebook):

$$V(a_4) = -2a_{12}a_4, \quad T(a_4) = -4a_{22}a_4 + a_{12}a_6, \quad 0 = 4a_{23}a_4 + a_{13}a_6, \quad (14)$$

$$W(a_4) = -2a_{13}a_4, \quad 0 = 4a_{32}a_4 + a_{13}a_6, \quad T(a_4) = -4a_{33}a_4 - a_{12}a_6, \quad (15)$$

$$T(a_6) - V(a_4) = 3a_{12}a_4 - a_{22}a_6, \quad 0 = a_{13}a_4 + (a_{23} + 3b_{13})a_6, \quad (16)$$

$$W(a_4) = (a_{23} + a_{32})a_6, \quad W(a_6) = (-a_{23} + 3a_{32})a_4 - b_{23}a_6, \quad (17)$$

$$V(a_6) = (-a_{22} + a_{33})a_4 + 3b_{33}a_6, \quad (18)$$

$$T(a_6) = -a_{12}a_4 - a_{33}a_6, \quad W(a_4) = -3a_{13}a_4 + (-a_{32} + 3b_{13})a_6, \quad (19)$$

$$V(a_4) = (-a_{22} + a_{33})a_6, \quad W(a_6) = (3a_{23} - a_{32})a_4 - 3b_{23}a_6, \quad (20)$$

$$0 = (a_{23} - a_{32})a_4, \quad (21)$$

$$W(a_4) = -a_{13}a_4 + (a_{32} - 3b_{13})a_6. \quad (21)$$

From the first equation of (15) (we will use the notation (15).1) and (17).1 resp. (14).3 and (15).2 we get:

$$0 = 2a_{13}a_4 + (a_{23} + a_{32})a_6, \quad 0 = 2(a_{23} + a_{32})a_4 + a_{13}a_6. \quad (22)$$

From (19).1) and (14).1 resp. (14).2 and (15).3 we get:

$$0 = -2a_{12}a_4 + 2(a_{22} - a_{33})a_6, \quad 0 = 2(-a_{22} + a_{33})a_4 + a_{12}a_6. \quad (23)$$

We consider first the case, that  $\mathbf{a}_6^2 \neq 4\mathbf{a}_4^2$ . Then we obtain from the foregoing equations that  $a_{13} = 0$ ,  $a_{32} = -a_{23}$ ,  $a_{12} = 0$  and  $a_{33} = a_{22}$ . Furthermore it follows from (14).1 that  $V(a_4) = 0$ , from (14).2 that  $T(a_4) = -4a_{22}a_4$  and from (14).3 that  $a_{23}a_4 = 0$ . Equation (15).1 becomes  $W(a_4) = 0$ , equation (16).2  $T(a_6) = -a_{22}a_6$  and (16).3  $(a_{23} + 3b_{13})a_6 = 0$ . Finally equation (17).2 resp. 3 gives  $W(a_6) = -3b_{23}a_6$  and  $V(a_6) = 3b_{33}a_6$ .

In case of  $SO(2)$ -symmetry ( $a_4 > 0$  and  $a_6 = 0$ ) it follows that  $a_{23} = 0$  and thus the statement of the theorem.

In case of  $S_3$ -symmetry ( $a_4 = 0$  and  $a_6 > 0$ ) it follows that  $a_{23} = -3b_{13}$  and thus the statement of the theorem.

In case of  $\mathbb{Z}_3$ -symmetry ( $a_4 > 0$  and  $a_6 > 0$ ) it follows that  $a_{23} = 0$  and  $b_{13} = 0$  and thus the statement of the theorem.

In case that  $a_6 = \pm 2a_4$  ( $\neq 0$ ), we can choose  $V$ ,  $W$  such that  $\mathbf{a}_6 = 2\mathbf{a}_4$ . Now equations (20), (14).3 and (16).3 lead to  $a_{23} = a_{32}$ ,  $a_{13} = -2a_{23}$  and  $b_{13} = 0$ . A combination of (14).2 and (15).3 gives  $a_{12} = (a_{22} - a_{33})$ , and then by equations (16).2, (14).1 and (14).2 that  $a_{33} = -a_{22}$ . Thus  $T(a_4) = 0$  by (14).2,  $V(a_4) = -4a_{22}a_4$  by (14).1 and  $W(a_4) = 4a_{22}a_4$  by (15).1. Finally (17).2 and (15).1 resp. (17).3 and (14).1 imply that  $b_{23} = -a_{23}$  resp.  $b_{33} = -a_{22}$ . ■

An evaluation of the Gauss equations (13) with the help of the CAS Mathematica leads to the following:

**Lemma 4.** *Let  $M^3$  be an affine hypersphere in  $\mathbb{R}^4$  which admits a pointwise  $\text{SO}(2)$ -,  $S_3$ - or  $\mathbb{Z}_3$ -symmetry and  $\{T, V, W\}$  the corresponding ONB. Then*

$$T(a_{22}) = -a_{22}^2 + a_{23}^2 + H - 3a_4^2, \quad (24)$$

$$T(a_{23}) = -2a_{22}a_{23}, \quad (25)$$

$$W(a_{22}) + V(a_{23}) = 0, \quad (26)$$

$$W(a_{23}) - V(a_{22}) = 0, \quad (27)$$

$$V(b_{13}) - T(b_{23}) = a_{22}b_{23} + (a_{23} + b_{13})b_{33}, \quad (28)$$

$$T(b_{33}) - W(b_{13}) = (a_{23} + b_{13})b_{23} - a_{22}b_{33}, \quad (29)$$

$$V(b_{33}) - W(b_{23}) = -a_{22}^2 - a_{23}^2 + 2a_{23}b_{13} + b_{23}^2 + b_{33}^2 + H + a_4^2 + 2a_6^2. \quad (30)$$

If the symmetry group is  $\mathbb{Z}_3$ , then  $a_6^2 \neq 4a_4^2$ .

**Proof.** The equations relate to eq11–eq13 and eq16 in the Mathematica notebook. If  $a_6^2 = 4a_4^2 (\neq 0)$ , then we obtain by equations eq11.1 and eq12.3 resp. eq15.3 and eq12.3 that  $2V(a_{22}) = -4a_{22}^2 - H + 3a_4^2$  resp.  $2W(a_{23}) = 4a_{23}^2 + H - 3a_4^2$ , thus  $V(a_{22}) - W(a_{23}) = -2a_{22}^2 - 2a_{23}^2 - H + 3a_4^2$ . This gives a contradiction to eq13.3, namely  $V(a_{22}) - W(a_{23}) = -2a_{22}^2 - 2a_{23}^2 - H - 9a_4^2$ . ■

## 5 Pointwise $\mathbb{Z}_3$ - or $\text{SO}(2)$ -symmetry

The following methods only work in the case of  $\mathbb{Z}_3$ - or  $\text{SO}(2)$ -symmetry, therefore the case of  $S_3$ -symmetry will be considered elsewhere. As the vector field  $T$  is globally defined, we can define the distributions  $L_1 = \text{span}\{T\}$  and  $L_2 = \text{span}\{V, W\}$ . In the following we will investigate these distributions. For the terminology we refer to [9].

**Lemma 5.** *The distribution  $L_1$  is autoparallel (totally geodesic) with respect to  $\widehat{\nabla}$ .*

**Proof.** From  $\widehat{\nabla}_T T = a_{12}V + a_{13}W = 0$  (cf. Lemmas 3 and 4) the claim follows immediately. ■

**Lemma 6.** *The distribution  $L_2$  is spherical with mean curvature normal  $U_2 = a_{22}T$ .*

**Proof.** For  $U_2 = a_{22}T \in L_1 = L_2^\perp$  we have  $h(\widehat{\nabla}_{E_a} E_b, T) = h(E_a, E_b)h(U_2, T)$  for  $E_a, E_b \in \{V, W\}$ , and  $h(\widehat{\nabla}_{E_a} U_2, T) = h(E_a(a_{22})T + a_{22}\widehat{\nabla}_{E_a} T, T) = 0$  (cf. Lemma 3 and (26), (27)). ■

**Remark 2.**  $a_{22}$  is independent of the choice of ONB  $\{V, W\}$ . It therefore is a globally defined function on  $M^3$ .

We introduce a coordinate function  $t$  by  $\frac{\partial}{\partial t} := T$ . Using the previous lemma, according to [12], we get:

**Lemma 7.**  *$(M^3, h)$  admits a warped product structure  $M^3 = I \times_{e^f} N^2$  with  $f : I \rightarrow \mathbb{R}$  satisfying*

$$\frac{\partial f}{\partial t} = a_{22}. \quad (31)$$

**Proof.** Proposition 3 in [12] gives the warped product structure with warping function  $\lambda_2 : I \rightarrow \mathbb{R}$ . If we introduce  $f = \ln \lambda_2$ , following the proof we see that  $a_{22}T = U_2 = -\text{grad}(\ln \lambda_2) = -\text{grad } f$ . ■

**Lemma 8.** *The curvature of  $N^2$  is  ${}^N K(N^2) = e^{2f}(H + 2a_6^2 + a_4^2 - a_{22}^2)$ .*



**Proof.** From Proposition 2 in [12] we get the following relation between the curvature tensor  $\hat{R}$  of the warped product  $M^3$  and the curvature tensor  $\tilde{R}$  of the usual product of pseudo-Riemannian manifolds ( $X, Y, Z \in \mathcal{X}(M)$  resp. their appropriate projections):

$$\begin{aligned}\hat{R}(X, Y)Z &= \tilde{R}(X, Y)Z + h(Y, Z)(\hat{\nabla}_X U_2 - h(X, U_2)U_2) - h(\hat{\nabla}_X U_2 - h(X, U_2)U_2, Z)Y \\ &\quad - h(X, Z)(\hat{\nabla}_Y U_2 - h(Y, U_2)U_2) + h(\hat{\nabla}_Y U_2 - h(Y, U_2)U_2, Z)X \\ &\quad + h(U_2, U_2)(h(Y, Z)X - h(X, Z)Y).\end{aligned}$$

Now  $\tilde{R}(X, Y)Z = {}^N\tilde{R}(X, Y)Z$  for all  $X, Y, Z \in TN^2$  and otherwise zero (cf. [11, page 89], Corollary 58) and  $K(N^2) = K(V, W) = \frac{h(-\hat{R}(V, W)V, W)}{h(V, V)h(W, W) - h(V, W)^2}$  (cf. [11, page 77], the curvature tensor has the opposite sign). Since  $h(X, Y) = e^{2f}h(X, Y)$  for  $X, Y \in TN^2$ , it follows that

$${}^N K(N^2) = e^{2f}h(-{}^N\hat{R}(V, W)V, W).$$

Finally we obtain by the Gauss equation (13) the last ingredient for the computation:  $\hat{R}(V, W)V = -(H + 2a_6^2 + a_4^2)W$  (cf. the Mathematica notebook). ■

Summarized we have obtained the following structure equations (cf. (1), (2) and (3)), where  $a_6 = 0$  in case of  $SO(2)$ -symmetry resp.  $b_{13} = 0$  in case of  $\mathbb{Z}_3$ -symmetry:

$$D_T T = -2a_4 T - \xi, \tag{32}$$

$$D_T V = +a_4 V - b_{13} W, \tag{33}$$

$$D_T W = +b_{13} V + a_4 W, \tag{34}$$

$$D_V T = +(a_{22} + a_4)V, \tag{35}$$

$$D_W T = +(a_{22} + a_4)W, \tag{36}$$

$$D_V V = +a_6 V - b_{23} W + (a_{22} - a_4)T + \xi, \tag{37}$$

$$D_V W = +b_{23} V - a_6 W, \tag{38}$$

$$D_W V = -(b_{33} + a_6)W, \tag{39}$$

$$D_W W = +(b_{33} - a_6)V + (a_{22} - a_4)T + \xi, \tag{40}$$

$$D_X \xi = -HX. \tag{41}$$

The Codazzi and Gauss equations ((12) and (13)) have the form (cf. Lemmas 3 and 4):

$$T(a_4) = -4a_{22}a_4, \quad 0 = V(a_4) = W(a_4), \tag{42}$$

$$T(a_6) = -a_{22}a_6, \quad V(a_6) = 3b_{33}a_6, \quad W(a_6) = -3b_{23}a_6, \tag{43}$$

$$T(a_{22}) = -a_{22}^2 + H - 3a_4^2, \quad V(a_{22}) = 0, \quad W(a_{22}) = 0, \tag{44}$$

$$V(b_{13}) - T(b_{23}) = a_{22}b_{23} + b_{13}b_{33}, \tag{45}$$

$$T(b_{33}) - W(b_{13}) = b_{13}b_{23} - a_{22}b_{33}, \tag{46}$$

$$V(b_{33}) - W(b_{23}) = -a_{22}^2 + b_{23}^2 + b_{33}^2 + H + a_4^2 + 2a_6^2, \tag{47}$$

where  $a_6 = 0$  in case of  $SO(2)$ -symmetry resp.  $b_{13} = 0$  in case of  $\mathbb{Z}_3$ -symmetry.

Our first goal is to find out how  $N^2$  is immersed in  $\mathbb{R}^4$ , i.e. to find an immersion independent of  $t$ . A look at the structure equations (32)–(41) suggests to start with a linear combination of  $T$  and  $\xi$ .

We will solve the problem in two steps. First we look for a vector field  $X$  with  $D_T X = \alpha X$  for some function  $\alpha$ : We define  $X := AT + \xi$  for some function  $A$  on  $M^3$ . Then  $D_T X = \alpha X$  iff  $\alpha = -A$  and  $\frac{\partial}{\partial t} A = -A^2 + 2a_4 A + H$ , and  $A := a_{22} - a_4$  solves the latter differential equation.



Next we want to multiply  $X$  with some function  $\beta$  such that  $D_T(\beta X) = 0$ : We define a positive function  $\beta$  on  $\mathbb{R}$  as the solution of the differential equation:

$$\frac{\partial}{\partial t}\beta = (a_{22} - a_4)\beta \quad (48)$$

with initial condition  $\beta(t_0) > 0$ . Then  $D_T(\beta X) = 0$  and by (35), (41) and (36) we get (since  $\beta$ ,  $a_{22}$  and  $a_4$  only depend on  $t$ ):

$$D_T(\beta((a_{22} - a_4)T + \xi)) = 0, \quad (49)$$

$$D_V(\beta((a_{22} - a_4)T + \xi)) = \beta(a_{22}^2 - a_4^2 - H)V, \quad (50)$$

$$D_W(\beta((a_{22} - a_4)T + \xi)) = \beta(a_{22}^2 - a_4^2 - H)W. \quad (51)$$

To obtain an immersion we need that  $\nu := a_{22}^2 - a_4^2 - H$  vanishes nowhere, but we only get:

**Lemma 9.** *The function  $\nu = a_{22}^2 - a_4^2 - H$  is globally defined,  $\frac{\partial}{\partial t}(e^{2f}\nu) = 0$  and  $\nu$  vanishes identically or nowhere on  $\mathbb{R}$ .*

**Proof.** Since  $0 = \frac{\partial}{\partial t}^N K(N^2) = \frac{\partial}{\partial t}(e^{2f}(2a_6^2 - \nu))$  (Lemma 8) and  $\frac{\partial}{\partial t}(e^{2f}2a_6^2) = 0$  (cf. (43) and (31)), we get that  $\frac{\partial}{\partial t}(e^{2f}\nu) = 0$ . Thus  $\frac{\partial}{\partial t}\nu = -2(\frac{\partial}{\partial t}f)\nu = -2a_{22}\nu$ . ■

### 5.1 The first case: $\nu \neq 0$ on $M^3$

We may, by translating  $f$ , i.e. by replacing  $N^2$  with a homothetic copy of itself, assume that  $e^{2f}\nu = \varepsilon_1$ , where  $\varepsilon_1 = \pm 1$ .

**Lemma 10.**  *$\Phi := \beta((a_{22} - a_4)T + \xi): M^3 \rightarrow \mathbb{R}^4$  induces a proper affine sphere structure, say  $\tilde{\phi}$ , mapping  $N^2$  into a 3-dimensional linear subspace of  $\mathbb{R}^4$ .  $\tilde{\phi}$  is part of a quadric iff  $a_6 = 0$ .*

**Proof.** By (50) and (51) we have  $\Phi_*(E_a) = \beta\nu E_a$  for  $E_a \in \{V, W\}$ . A further differentiation, using (37) ( $\beta$  and  $\nu$  only depend on  $t$ ), gives:

$$\begin{aligned} D_V\Phi_*(V) &= \beta\nu D_VV = \beta\nu((a_{22} - a_4)T + a_6V - b_{23}W + \xi) \\ &= a_6\Phi_*(V) - b_{23}\Phi_*(W) + \nu\Phi = a_6\Phi_*(V) - b_{23}\Phi_*(W) + \varepsilon_1 e^{-2f}\Phi. \end{aligned}$$

Similarly, we obtain the other derivatives, using (38)–(40), thus:

$$D_V\Phi_*(V) = a_6\Phi_*(V) - b_{23}\Phi_*(W) + e^{-2f}\varepsilon_1\Phi, \quad (52)$$

$$D_V\Phi_*(W) = b_{23}\Phi_*(V) - a_6\Phi_*(W), \quad (53)$$

$$D_W\Phi_*(V) = -(b_{33} + a_6)\Phi_*(W), \quad (54)$$

$$D_W\Phi_*(W) = (b_{33} - a_6)\Phi_*(V) + e^{-2f}\varepsilon_1\Phi, \quad (55)$$

$$D_{E_a}\Phi = \beta e^{-2f}\varepsilon_1 E_a. \quad (56)$$

The foliation at  $f = f_0$  gives an immersion of  $N^2$  to  $M^3$ , say  $\pi_{f_0}$ . Therefore, we can define an immersion of  $N^2$  to  $\mathbb{R}^4$  by  $\tilde{\phi} := \Phi \circ \pi_{f_0}$ , whose structure equations are exactly the equations above when  $f = f_0$ . Hence, we know that  $\tilde{\phi}$  maps  $N^2$  into  $\text{span}\{\Phi_*(V), \Phi_*(W), \Phi\}$ , an affine hyperplane of  $\mathbb{R}^4$  and  $\frac{\partial}{\partial t}\Phi = 0$  implies  $\Phi(t, v, w) = \tilde{\phi}(v, w)$ .

We can read off the coefficients of the difference tensor  $K^{\tilde{\phi}}$  of  $\tilde{\phi}$  (cf. (1) and (3)):  $K^{\tilde{\phi}}(\tilde{V}, \tilde{V}) = a_6\tilde{V}$ ,  $K^{\tilde{\phi}}(\tilde{V}, \tilde{W}) = -a_6\tilde{W}$ ,  $K^{\tilde{\phi}}(\tilde{W}, \tilde{W}) = -a_6\tilde{V}$ , and see that  $\text{trace}(K^{\tilde{\phi}})_X$  vanishes. The affine metric introduced by this immersion corresponds with the metric on  $N^2$ . Thus  $\varepsilon_1\tilde{\phi}$  is the affine normal of  $\tilde{\phi}$  and  $\tilde{\phi}$  is a proper affine sphere with mean curvature  $\varepsilon_1$ . Finally the vanishing of the difference tensor characterizes quadrics. ■

Our next goal is to find another linear combination of  $T$  and  $\xi$ , this time only depending on  $t$ . (Then we can express  $T$  in terms of  $\phi$  and some function of  $t$ .)

**Lemma 11.** *Define  $\delta := HT + (a_{22} + a_4)\xi$ . Then there exist a constant vector  $C \in \mathbb{R}^4$  and a function  $a(t)$  such that*

$$\delta(t) = a(t)C.$$

**Proof.** Using (35) resp. (36) and (41) we obtain that  $D_V\delta = 0 = D_W\delta$ . Hence  $\delta$  depends only on the variable  $t$ . Moreover, we get by (32), (44), (42) and (41) that

$$\begin{aligned} \frac{\partial}{\partial t}\delta &= D_T(HT + (a_{22} + a_4)\xi) \\ &= H(-2a_4T - \xi) + (-a_{22}^2 + H - 3a_4^2 - 4a_{22}a_4)\xi - (a_{22} + a_4)HT \\ &= -(3a_4 + a_{22})(HT + (a_{22} + a_4)\xi) = -(3a_4 + a_{22})\delta. \end{aligned}$$

This implies that there exists a constant vector  $C$  in  $\mathbb{R}^4$  and a function  $a(t)$  such that  $\delta(t) = a(t)C$ .  $\blacksquare$

Notice that for an improper affine hypersphere ( $H = 0$ )  $\xi$  is constant and parallel to  $C$ . Combining  $\tilde{\phi}$  and  $\delta$  we obtain for  $T$  (cf. Lemmas 10 and 11) that

$$T(t, v, w) = -\frac{a}{\nu}C + \frac{1}{\beta\nu}(a_{22} + a_4)\tilde{\phi}(v, w). \quad (57)$$

In the following we will use for the partial derivatives the abbreviation  $\varphi_x := \frac{\partial}{\partial x}\varphi$ ,  $x = t, v, w$ .

**Lemma 12.**

$$\varphi_t = -\frac{a}{\nu}C + \frac{\partial}{\partial t}\left(\frac{1}{\beta\nu}\right)\tilde{\phi}, \quad \varphi_v = \frac{1}{\beta\nu}\tilde{\phi}_v, \quad \varphi_w = \frac{1}{\beta\nu}\tilde{\phi}_w.$$

**Proof.** As by (48) and Lemma 9  $\frac{\partial}{\partial t}\frac{1}{\beta\nu} = \frac{1}{\beta\nu}(a_{22} + a_4)$ , we obtain the equation for  $\varphi_t = T$  by (57). The other equations follow from (50) and (51).  $\blacksquare$

It follows by the uniqueness theorem of first order differential equations and applying a translation that we can write

$$\varphi(t, v, w) = \tilde{a}(t)C + \frac{1}{\beta\nu}(t)\tilde{\phi}(v, w)$$

for a suitable function  $\tilde{a}$  depending only on the variable  $t$ . Since  $C$  is transversal to the image of  $\tilde{\phi}$  (cf. Lemmas 10 and 11,  $\nu \neq 0$ ), we obtain that after applying an equiaffine transformation we can write:  $\varphi(t, v, w) = (\gamma_1(t), \gamma_2(t)\phi(v, w))$ , in which  $\tilde{\phi}(v, w) = (0, \phi(v, w))$ . Thus we have proven the following:

**Theorem 1.** *Let  $M^3$  be an indefinite affine hypersphere of  $\mathbb{R}^4$  which admits a pointwise  $\mathbb{Z}_3$ - or  $SO(2)$ -symmetry. Let  $a_{22}^2 - a_4^2 \neq H$  for some  $p \in M^3$ . Then  $M^3$  is affine equivalent to*

$$\varphi : I \times N^2 \rightarrow \mathbb{R}^4 : (t, v, w) \mapsto (\gamma_1(t), \gamma_2(t)\phi(v, w)),$$

where  $\phi : N^2 \rightarrow \mathbb{R}^3$  is a (positive definite) elliptic or hyperbolic affine sphere and  $\gamma : I \rightarrow \mathbb{R}^2$  is a curve. Moreover, if  $M^3$  admits a pointwise  $SO(2)$ -symmetry then  $N^2$  is either an ellipsoid or a two-sheeted hyperboloid.

We want to investigate the conditions imposed on the curve  $\gamma$ . For this we compute the derivatives of  $\varphi$ :

$$\begin{aligned}\varphi_t &= (\gamma'_1, \gamma'_2 \phi), & \varphi_v &= (0, \gamma_2 \phi_v), & \varphi_w &= (0, \gamma_2 \phi_w), \\ \varphi_{tt} &= (\gamma''_1, \gamma''_2 \phi), & \varphi_{tv} &= (0, \gamma'_2 \phi_v), & \varphi_{tw} &= (0, \gamma'_2 \phi_w), \\ \varphi_{vv} &= (0, \gamma_2 \phi_{vv}), & \varphi_{vw} &= (0, \gamma_2 \phi_{vw}), & \varphi_{ww} &= (0, \gamma'_2 \phi_{ww}).\end{aligned}\tag{58}$$

Furthermore we have to distinguish if  $M^3$  is proper ( $H = \pm 1$ ) or improper ( $H = 0$ ).

First we consider the case that  $M^3$  is proper, i.e.  $\xi = -H\varphi$ . An easy computation shows that the condition that  $\xi$  is a transversal vector field, namely  $0 \neq \det(\varphi_t, \varphi_v, \varphi_w, \xi) = -\gamma_2^2(\gamma_1 \gamma'_2 - \gamma'_1 \gamma_2) \det(\phi_v, \phi_w, \phi)$ , is equivalent to  $\gamma_2 \neq 0$  and  $\gamma_1 \gamma'_2 - \gamma'_1 \gamma_2 \neq 0$ . To check the condition that  $\xi$  is the Blaschke normal (cf. (4)), we need to compute the Blaschke metric  $h$ , using (1), (58), (52)–(55) and the notation  $r, s \in \{v, w\}$  and  $g$  for the Blaschke metric of  $\phi$ :

$$\begin{aligned}\varphi_{tt} &= \cdots \varphi_t + \frac{\gamma'_1 \gamma''_2 - \gamma''_1 \gamma'_2}{H(\gamma_1 \gamma'_2 - \gamma'_1 \gamma_2)} \xi, & \varphi_{tr} &= \text{tang}, \\ \varphi_{rs} &= \text{tang} - \frac{\gamma'_1 \gamma_2}{H(\gamma_1 \gamma'_2 - \gamma'_1 \gamma_2)} \varepsilon_1 g \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial s} \right) \xi.\end{aligned}$$

We obtain that

$$\det h = h_{tt}(h_{vv}h_{ww} - h_{vw}^2) = \frac{\gamma'_1 \gamma''_2 - \gamma''_1 \gamma'_2}{H^3(\gamma_1 \gamma'_2 - \gamma'_1 \gamma_2)^3} (\gamma'_1)^2 \gamma_2^2 \det g.$$

Thus

$$\gamma_2^4 (\gamma_1 \gamma'_2 - \gamma'_1 \gamma_2)^2 \det(\phi_v, \phi_w, \phi)^2 = \left| \frac{\gamma'_1 \gamma''_2 - \gamma''_1 \gamma'_2}{(\gamma_1 \gamma'_2 - \gamma'_1 \gamma_2)^3} (\gamma'_1)^2 \gamma_2^2 \det g \right|$$

is equivalent to (4). Since  $\phi$  is a definite proper affine sphere with normal  $-\varepsilon_1 \phi$ , we can again use (4) to obtain

$$\xi = -H\varphi \iff \gamma_2^2 |\gamma_1 \gamma'_2 - \gamma'_1 \gamma_2|^5 = |\gamma'_1 \gamma''_2 - \gamma''_1 \gamma'_2| (\gamma'_1)^2 \neq 0.$$

From the computations above ( $g$  is positive definite) also it follows that  $\varphi$  is indefinite iff either

$$\begin{aligned}H \operatorname{sign}(\gamma_1 \gamma'_2 - \gamma'_1 \gamma_2) &= \operatorname{sign}(\gamma'_1 \gamma''_2 - \gamma''_1 \gamma'_2) = \operatorname{sign}(\gamma'_1 \gamma_2 \varepsilon_1) & \text{or} \\ -H \operatorname{sign}(\gamma_1 \gamma'_2 - \gamma'_1 \gamma_2) &= \operatorname{sign}(\gamma'_1 \gamma''_2 - \gamma''_1 \gamma'_2) = \operatorname{sign}(\gamma'_1 \gamma_2 \varepsilon_1).\end{aligned}$$

Next we consider the case that  $M^3$  is improper, i.e.  $\xi$  is constant. By Lemma 11  $\xi$  is parallel to  $C$  and thus transversal to  $\phi$ . Hence we can apply an affine transformation to obtain  $\xi = (1, 0, 0, 0)$ . An easy computation shows that the condition that  $\xi$  is a transversal vector field, namely  $0 \neq \det(\varphi_t, \varphi_v, \varphi_w, \xi) = -\gamma_2^2 \gamma'_2 \det(\phi_v, \phi_w, \phi)$ , is equivalent to  $\gamma_2 \neq 0$  and  $\gamma'_2 \neq 0$ . To check the condition that  $\xi$  is the Blaschke normal (cf. (4)) we need to compute the Blaschke metric  $h$ , using (1), (58), (52)–(55) and the notation  $r, s \in \{v, w\}$  and  $g$  for the Blaschke metric of  $\phi$ :

$$\varphi_{tt} = \cdots \varphi_t - \frac{\gamma'_1 \gamma''_2 - \gamma''_1 \gamma'_2}{\gamma'_2} \xi, \quad \varphi_{tr} = \text{tang}, \quad \varphi_{rs} = \text{tang} + \frac{\gamma'_1 \gamma_2}{\gamma'_2} \varepsilon_1 g \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial s} \right) \xi.$$

We obtain that

$$\det h = h_{tt}(h_{vv}h_{ww} - h_{vw}^2) = -\frac{\gamma'_1 \gamma''_2 - \gamma''_1 \gamma'_2}{(\gamma'_2)^3} (\gamma'_1)^2 \gamma_2^2 \det g.$$

Thus (4) is equivalent to

$$\gamma_2^4(\gamma_2')^2 \det(\phi_v, \phi_w, \phi)^2 = \left| \frac{\gamma_1' \gamma_2'' - \gamma_1'' \gamma_2'}{(\gamma_2')^3} (\gamma_1')^2 \gamma_2^2 \det g \right|.$$

Since  $\phi$  is a definite proper affine sphere with normal  $-\varepsilon_1 \phi$ , we can again use (4) to obtain

$$\xi = (1, 0, 0, 0) \iff \gamma_2^2 |\gamma_2'|^5 = |\gamma_1' \gamma_2'' - \gamma_1'' \gamma_2'| (\gamma_1')^2 \neq 0.$$

From the computations above also it follows that  $\varphi$  is indefinite iff either

$$\begin{aligned} -\operatorname{sign}(\gamma_2') &= \operatorname{sign}(\gamma_1' \gamma_2'' - \gamma_1'' \gamma_2') = \operatorname{sign}(\gamma_1' \gamma_2 \varepsilon_1) \quad \text{or} \\ \operatorname{sign}(\gamma_2') &= \operatorname{sign}(\gamma_1' \gamma_2'' - \gamma_1'' \gamma_2') = \operatorname{sign}(\gamma_1' \gamma_2 \varepsilon_1). \end{aligned}$$

So we have seen under which conditions we can construct a 3-dimensional indefinite affine hypersphere out of an affine sphere:

**Theorem 2.** *Let  $\phi : N^2 \rightarrow \mathbb{R}^3$  be a positive definite elliptic or hyperbolic affine sphere (with mean curvature  $\varepsilon_1 = \pm 1$ ), and let  $\gamma = (\gamma_1, \gamma_2) : I \rightarrow \mathbb{R}^2$  be a curve. Define  $\varphi : I \times N^2 \rightarrow \mathbb{R}^4$  by  $\varphi(t, v, w) = (\gamma_1(t), \gamma_2(t), \phi(v, w))$ .*

- (i) *If  $\gamma$  satisfies  $\gamma_2^2 |\gamma_1' \gamma_2'' - \gamma_1'' \gamma_2'|^5 = \operatorname{sign}(\gamma_1' \gamma_2 \varepsilon_1) (\gamma_1' \gamma_2'' - \gamma_1'' \gamma_2') (\gamma_1')^2 \neq 0$ , then  $\varphi$  defines a 3-dimensional indefinite proper affine hypersphere.*
- (ii) *If  $\gamma$  satisfies  $\gamma_2^2 |\gamma_2'|^5 = \operatorname{sign}(\gamma_1' \gamma_2 \varepsilon_1) (\gamma_1' \gamma_2'' - \gamma_1'' \gamma_2') (\gamma_1')^2 \neq 0$ , then  $\varphi$  defines a 3-dimensional indefinite improper affine hypersphere.*

Now we are ready to check the symmetries.

**Theorem 3.** *Let  $\phi : N^2 \rightarrow \mathbb{R}^3$  be a positive definite elliptic or hyperbolic affine sphere (with mean curvature  $\varepsilon_1 = \pm 1$ ), and let  $\gamma = (\gamma_1, \gamma_2) : I \rightarrow \mathbb{R}^2$  be a curve such that  $\varphi(t, v, w) = (\gamma_1(t), \gamma_2(t), \phi(v, w))$  defines a 3-dimensional indefinite affine hypersphere. Then  $\varphi(N^2 \times I)$  admits a pointwise  $\mathbb{Z}_3$ - or  $SO(2)$ -symmetry.*

**Proof.** We already have shown that  $\varphi$  defines a 3-dimensional indefinite proper resp. improper affine hypersphere. To prove the symmetry we need to compute  $K$ . By assumption,  $\phi$  is an affine sphere with Blaschke normal  $\xi^\phi = -\varepsilon_1 \phi$ . For the structure equations (1) we use the notation  $\phi_{rs} = \phi \Gamma_{rs}^u \phi_u - g_{rs} \varepsilon_1 \phi$ ,  $r, s, u \in \{v, w\}$ . Furthermore we introduce the notation  $\alpha = \gamma_1 \gamma_2' - \gamma_1' \gamma_2$ . Note that  $\alpha' = \gamma_1 \gamma_2'' - \gamma_1'' \gamma_2$ . If  $\varphi$  is proper, using (58), we get the structure equations (1) for  $\varphi$ :

$$\begin{aligned} \varphi_{tt} &= \frac{\alpha'}{\alpha} \varphi_t + \frac{\gamma_1' \gamma_2'' - \gamma_1'' \gamma_2'}{H \alpha} \xi, & \varphi_{tr} &= \frac{\gamma_2'}{\gamma_2} \varphi_r, \\ \varphi_{rs} &= \phi \Gamma_{rs}^u \phi_u - g_{rs} \varepsilon_1 \frac{\gamma_1 \gamma_2}{\alpha} \varphi_t - g_{rs} \varepsilon_1 \frac{\gamma_1' \gamma_2'}{H \alpha} \xi. \end{aligned}$$

We compute  $K$  using (6) and obtain:

$$\begin{aligned} (\nabla_{\varphi_t} h)(\varphi_r, \varphi_s) &= \left( \left( \frac{\gamma_1 \gamma_2}{\alpha} \right)' \frac{\alpha}{\gamma_1 \gamma_2} - 2 \frac{\gamma_2'}{\gamma_2} \right) h(\varphi_r, \varphi_s), \\ (\nabla_{\varphi_r} h)(\varphi_t, \varphi_t) &= 0, \end{aligned}$$

implying that  $K_{\varphi_t}$  restricted to the space spanned by  $\varphi_v$  and  $\varphi_w$  is a multiple of the identity. Taking  $T$  in direction of  $\varphi_t$ , we see that  $\varphi_v$  and  $\varphi_w$  are orthogonal to  $T$ . Thus we can construct an ONB  $\{T, V, W\}$  with  $V, W$  spanning  $\operatorname{span}\{\varphi_v, \varphi_w\}$  such that  $a_1 = 2a_4$ ,  $a_2 = a_3 = a_5 = 0$ . By the considerations in [13, Section 4] we see that  $\varphi$  admits a pointwise  $\mathbb{Z}_3$ - or  $SO(2)$ -symmetry. If  $\varphi$  is improper, the proof runs completely analogous.  $\blacksquare$

## 5.2 The second case: $\nu \equiv 0$ and $H \neq 0$ on $M^3$

Next, we consider the case that  $H = a_{22}^2 - a_4^2$  and  $H \neq 0$  on  $M^3$ . It follows that  $a_{22} \neq \pm a_4$  on  $M^3$ .

We already have seen that  $M^3$  admits a warped product structure. The map  $\Phi$  we have constructed in Lemma 10 will not define an immersion (cf. (50) and (51)). Anyhow, for a fixed point  $t_0$ , we get from (37)–(40), (50) and (51), using the notation  $\tilde{\xi} = (a_{22} - a_4)T + \xi$ :

$$\begin{aligned} D_V V &= a_6 V - b_{23} W + \tilde{\xi}, & D_V W &= b_{23} V - a_6 W, \\ D_W V &= -(b_{33} + a_6)W, & D_W W &= (b_{33} - a_6)V + \tilde{\xi}, & D_{E_a} \tilde{\xi} &= 0, & E_a &\in \{V, W\}. \end{aligned}$$

Thus, if  $v$  and  $w$  are local coordinates which span the second distribution  $L_2$ , then we can interpret  $\varphi(t_0, v, w)$  as a positive definite improper affine sphere in a 3-dimensional linear subspace.

Moreover, we see that this improper affine sphere is a paraboloid provided that  $a_6(t_0, v, w)$  vanishes identically. From the differential equations (43) determining  $a_6$ , we see that this is the case exactly when  $a_6$  vanishes identically, i.e. when  $M^3$  admits a pointwise  $SO(2)$ -symmetry.

After applying a translation and a change of coordinates, we may assume that

$$\varphi(t_0, v, w) = (v, w, f(v, w), 0),$$

with affine normal  $\tilde{\xi}(t_0, v, w) = (0, 0, 1, 0)$ . To obtain  $T$  at  $t_0$ , we consider (35) and (36) and get that

$$D_{E_a}(T - (a_{22} + a_4)\varphi) = 0, \quad E_a, E_b \in \{V, W\}.$$

Evaluating at  $t = t_0$ , this means that there exists a constant vector  $C$ , transversal to  $\text{span}\{V, W, \xi\}$ , such that  $T(t_0, v, w) = (a_{22} + a_4)(t_0)\varphi(t_0, v, w) + C$ . Since  $a_{22} + a_4 \neq 0$  everywhere, we can write:

$$T(t_0, v, w) = \alpha_1(v, w, f(v, w), \alpha_2), \tag{59}$$

where  $\alpha_1, \alpha_2 \neq 0$  and we applied an equiaffine transformation so that  $C = (0, 0, 0, \alpha_1 \alpha_2)$ . To obtain information about  $D_T T$  we have that  $D_T T = -2a_4 T - \xi$  (cf. (32)) and  $\xi = \tilde{\xi} - (a_{22} - a_4)T$  by the definition of  $\tilde{\xi}$ . Also we know that  $\tilde{\xi}(t_0, v, w) = (0, 0, 1, 0)$  and by (49)–(51) that  $D_X(\beta\tilde{\xi}) = 0$ ,  $X \in \mathcal{X}(M)$ . Taking suitable initial conditions for the function  $\beta$  ( $\beta(t_0) = 1$ ), we get that  $\beta\tilde{\xi} = (0, 0, 1, 0)$  and finally the following vector valued differential equation:

$$D_T T = (a_{22} - 3a_4)T - \frac{1}{\beta}(0, 0, 1, 0).$$

Solving this differential equation, taking into account the initial conditions (59) at  $t = t_0$ , we get that there exist functions  $\delta_1$  and  $\delta_2$  depending only on  $t$  such that

$$T(t, u, v) = (\delta_1(t)v, \delta_1(t)w, \delta_1(t)(f(v, w) + \delta_2(t)), \alpha_2 \delta_1(t)),$$

where  $\delta_1(t_0) = \alpha_1$ ,  $\delta_2(t_0) = 0$ ,  $\delta_1'(t) = (a_{22} - 3a_4)\delta_1(t)$  and  $\delta_2'(t) = \delta_1^{-1}(t)\beta^{-1}(t)$ . As  $T(t, v, w) = \frac{\partial \varphi}{\partial t}(t, v, w)$  and  $\varphi(t_0, v, w) = (v, w, f(v, w), 0)$  it follows by integration that

$$\varphi(t, v, w) = (\gamma_1(t)v, \gamma_1(t)w, \gamma_1(t)f(v, w) + \gamma_2(t), \alpha_2(\gamma_1(t) - 1)),$$

where  $\gamma_1'(t) = \delta_1(t)$ ,  $\gamma_1(t_0) = 1$ ,  $\gamma_2(t_0) = 0$  and  $\gamma_2'(t) = \delta_1(t)\delta_2(t)$ . After applying an affine transformation we have shown:

**Theorem 4.** *Let  $M^3$  be an indefinite proper affine hypersphere of  $\mathbb{R}^4$  which admits a pointwise  $\mathbb{Z}_3$ - or  $SO(2)$ -symmetry. Let  $H = a_{22}^2 - a_4^2 (\neq 0)$  on  $M^3$ . Then  $M^3$  is affine equivalent with*

$$\varphi : I \times N^2 \rightarrow \mathbb{R}^4 : (t, v, w) \mapsto (\gamma_1(t)v, \gamma_1(t)w, \gamma_1(t)f(v, w) + \gamma_2(t), \gamma_1(t)),$$

where  $\psi : N^2 \rightarrow \mathbb{R}^3 : (v, w) \mapsto (v, w, f(v, w))$  is a positive definite improper affine sphere with affine normal  $(0, 0, 1)$  and  $\gamma : I \rightarrow \mathbb{R}^2$  is a curve. Moreover, if  $M^3$  admits a pointwise  $SO(2)$ -symmetry then  $N^2$  is an elliptic paraboloid.

We want to investigate the conditions imposed on the curve  $\gamma$ . For this we compute the derivatives of  $\varphi$ :

$$\begin{aligned} \varphi_t &= (\gamma_1'v, \gamma_1'w, \gamma_1'f(v, w) + \gamma_2', \gamma_1'), \\ \varphi_v &= (\gamma_1, 0, \gamma_1f_v, 0), \quad \varphi_w = (0, \gamma_1, \gamma_1f_w, 0), \\ \varphi_{tt} &= (\gamma_1''v, \gamma_1''w, \gamma_1''f(v, w) + \gamma_2'', \gamma_1''), \\ \varphi_{tv} &= \frac{\gamma_1'}{\gamma_1}\varphi_v, \quad \varphi_{tw} = \frac{\gamma_1'}{\gamma_1}\varphi_w, \\ \varphi_{vv} &= (0, 0, f_{vv}\gamma_1, 0), \quad \varphi_{vw} = (0, 0, \gamma_1f_{vw}, 0), \quad \varphi_{ww} = (0, 0, \gamma_1f_{ww}, 0). \end{aligned} \tag{60}$$

$M^3$  is a proper hypersphere, i.e.  $\xi = -H\varphi$ . An easy computation shows that the condition that  $\xi$  is a transversal vector field, namely  $0 \neq \det(\varphi_t, \varphi_v, \varphi_w, \xi) = -H\gamma_1^2(\gamma_1\gamma_2' - \gamma_1'\gamma_2)$ , is equivalent to  $\gamma_1 \neq 0$  and  $\gamma_1\gamma_2' - \gamma_1'\gamma_2 \neq 0$ . Since  $(0, 0, 1, 0) = \frac{\gamma_1}{\gamma_1\gamma_2' - \gamma_1'\gamma_2}\varphi_t - \frac{\gamma_1'}{\gamma_1\gamma_2' - \gamma_1'\gamma_2}\varphi$ , we have the following structure equations:

$$\begin{aligned} \varphi_{tt} &= \left( \frac{\gamma_1''}{\gamma_1} + \frac{\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'}{\gamma_1} \frac{\gamma_1}{\gamma_1\gamma_2' - \gamma_1'\gamma_2} \right) \varphi_t + \frac{\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'}{\gamma_1\gamma_2' - \gamma_1'\gamma_2} \frac{1}{H} \xi, \\ \varphi_{tr} &= \frac{\gamma_1'}{\gamma_1}\varphi_r, \quad \varphi_{rs} = \frac{\gamma_1^2}{\gamma_1\gamma_2' - \gamma_1'\gamma_2} f_{rs}\varphi_t + \frac{\gamma_1\gamma_1'}{\gamma_1\gamma_2' - \gamma_1'\gamma_2} f_{rs} \frac{1}{H} \xi. \end{aligned} \tag{61}$$

We obtain:

$$\det h = h_{tt}(h_{vv}h_{ww} - h_{vw}^2) = \frac{\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'}{H^3(\gamma_1\gamma_2' - \gamma_1'\gamma_2)^3} \gamma_1^2(\gamma_1')^2(f_{vv}f_{ww} - f_{vw}^2).$$

Since  $\psi$  is a positive definite improper affine sphere with affine normal  $(0, 0, 1)$ , we get by (4) that  $f_{vv}f_{ww} - f_{vw}^2 = 1$ . Now (4) (for  $\xi$ ) is equivalent to

$$\gamma_1^4(\gamma_1\gamma_2' - \gamma_1'\gamma_2)^2 = \left| \frac{\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'}{(\gamma_1\gamma_2' - \gamma_1'\gamma_2)^3} \right| \gamma_1^2(\gamma_1')^2.$$

It follows that

$$\xi = -H\varphi \iff \gamma_1^2|\gamma_1\gamma_2' - \gamma_1'\gamma_2|^5 = |\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'|(\gamma_1')^2 \neq 0.$$

From the computations above also it follows that  $\varphi$  is indefinite iff either

$$\begin{aligned} \text{sign}(\gamma_1'\gamma_2'' - \gamma_1''\gamma_2') &= \text{sign}(H(\gamma_1\gamma_2' - \gamma_1'\gamma_2)) = -\text{sign}(\gamma_1\gamma_1') \quad \text{or} \\ \text{sign}(\gamma_1'\gamma_2'' - \gamma_1''\gamma_2') &= -\text{sign}(H(\gamma_1\gamma_2' - \gamma_1'\gamma_2)) = -\text{sign}(\gamma_1\gamma_1'). \end{aligned}$$

So we have seen under which conditions we can construct a 3-dimensional indefinite affine hypersphere out of an affine sphere:

**Theorem 5.** *Let  $\psi : N^2 \rightarrow \mathbb{R}^3 : (v, w) \mapsto (v, w, f(v, w))$  be a positive definite improper affine sphere with affine normal  $(0, 0, 1)$ , and let  $\gamma : I \rightarrow \mathbb{R}^2$  be a curve. Define  $\varphi : I \times N^2 \rightarrow \mathbb{R}^4$  by  $\varphi(t, v, w) = (\gamma_1(t)v, \gamma_1(t)w, \gamma_1(t)f(v, w) + \gamma_2(t), \gamma_1(t))$ . If  $\gamma = (\gamma_1, \gamma_2)$  satisfies  $\gamma_1^2|\gamma_1\gamma_2' - \gamma_1'\gamma_2|^5 = -\text{sign}(\gamma_1\gamma_1')(\gamma_1'\gamma_2'' - \gamma_1''\gamma_2')(\gamma_1')^2 \neq 0$ , then  $\varphi$  defines a 3-dimensional indefinite proper affine hypersphere.*

Now we are ready to check the symmetries.

**Theorem 6.** *Let  $\psi : N^2 \rightarrow \mathbb{R}^3 : (v, w) \mapsto (v, w, f(v, w))$  be a positive definite improper affine sphere with affine normal  $(0, 0, 1)$ , and let  $\gamma : I \rightarrow \mathbb{R}^2$  be a curve such that  $\varphi(t, v, w) = (\gamma_1(t)v, \gamma_1(t)w, \gamma_1(t)f(v, w) + \gamma_2(t), \gamma_1(t))$  defines a 3-dimensional indefinite proper affine hypersphere. Then  $\varphi(N^2 \times I)$  admits a pointwise  $\mathbb{Z}_3$ - or  $SO(2)$ -symmetry.*

**Proof.** We already have shown that  $\varphi$  defines a 3-dimensional indefinite proper affine hypersphere with affine normal  $\xi = -H\varphi$ . To prove the symmetry we need to compute  $K$ . We get the induced connection and the affine metric from the structure equations (61). We compute  $K$  using (6) and obtain:

$$\begin{aligned} (\nabla_{\varphi_t} h)(\varphi_r, \varphi_s) &= \left( \frac{\partial}{\partial t} \ln \left( \frac{\gamma_1 \gamma_1'}{\gamma_1 \gamma_2' - \gamma_1' \gamma_2} \right) - 2 \frac{\gamma_1'}{\gamma_1} \right) h(\varphi_r, \varphi_s), \\ (\nabla_{\varphi_r} h)(\varphi_t, \varphi_t) &= 0, \end{aligned}$$

implying that  $K_{\varphi_t}$  restricted to the space spanned by  $\varphi_v$  and  $\varphi_w$  is a multiple of the identity. Taking  $T$  in direction of  $\varphi_t$ , we see that  $\varphi_v$  and  $\varphi_w$  are orthogonal to  $T$ . Thus we can construct an ONB  $\{T, V, W\}$  with  $V, W$  spanning  $\text{span}\{\varphi_v, \varphi_w\}$  such that  $a_1 = 2a_4$ ,  $a_2 = a_3 = a_5 = 0$ . By the considerations in [13, Section 4] we see that  $\varphi$  admits a pointwise  $\mathbb{Z}_3$ - or  $SO(2)$ -symmetry. ■

### 5.3 The third case: $\nu \equiv 0$ and $H = 0$ on $M^3$

The final cases now are that  $\nu \equiv 0$  and  $H = 0$  on the whole of  $M^3$  and hence  $a_{22} = \pm a_4$ .

First we consider the case that  $a_{22} = a_4 =: a > 0$ . Again we use that  $M^3$  admits a warped product structure and we fix a parameter  $t_0$ . At the point  $t_0$ , we have by (37)–(41):

$$\begin{aligned} D_V V &= +a_6 V - b_{23} W + \xi, \\ D_V W &= +b_{23} V - a_6 W, \\ D_W V &= -(b_{33} + a_6) W, \\ D_W W &= +(b_{33} - a_6) V + \xi, \\ D_X \xi &= 0. \end{aligned}$$

Thus, if  $v$  and  $w$  are local coordinates which span the second distribution  $L_2$ , then we can interpret  $\varphi(t_0, v, w)$  as a positive definite improper affine sphere in a 3-dimensional linear subspace.

Moreover, we see that this improper affine sphere is a paraboloid provided that  $a_6(t_0, v, w)$  vanishes identically. From the differential equations (43) determining  $a_6$ , we see that this is the case exactly when  $a_6$  vanishes identically, i.e. when  $M^3$  admits a pointwise  $SO(2)$ -symmetry.

After applying an affine transformation and a change of coordinates, we may assume that

$$\varphi(t_0, v, w) = (v, w, f(v, w), 0), \tag{62}$$

with affine normal  $\xi(t_0, v, w) = (0, 0, 1, 0)$ , actually

$$\xi(t, v, w) = (0, 0, 1, 0)$$

( $\xi$  is constant on  $M^3$  by assumption). Furthermore we obtain by (35) and (36), that  $D_U T = 2aU$  for all  $U \in L_2$ . We define  $\delta := T - 2a\varphi$ , which is transversal to  $\text{span}\{V, W, \xi\}$ . Since  $a$  is independent of  $v$  and  $w$  (cf. (42)),  $D_U \delta = 0$ , and we can assume that

$$T(t_0, v, w) - 2a(t_0)\varphi(t_0, v, w) = (0, 0, 0, 1). \tag{63}$$



We can integrate (42) ( $T(a) = -4a^2$ ) and we take  $a = \frac{1}{4t}$ ,  $t > 0$ . Thus (32) becomes  $D_T T = -\frac{1}{2t}T - \xi$  and we obtain the following linear second order ordinary differential equation:

$$\frac{\partial^2}{\partial t^2}\varphi + \frac{1}{2t}\frac{\partial}{\partial t}\varphi = -\xi. \quad (64)$$

The general solution is  $\varphi(t, v, w) = -\frac{t^2}{3}\xi + 2\sqrt{t}A(v, w) + B(v, w)$ . The initial conditions (62) and (63) imply that  $A(v, w) = (\frac{v}{2\sqrt{t_0}}, \frac{w}{2\sqrt{t_0}}, \frac{f(v, w)}{2\sqrt{t_0}} + \frac{2}{3}t_0^{3/2}, \sqrt{t_0})$  and  $B(v, w) = (0, 0, -t_0^2, -2t_0)$ . Obviously we can translate  $B$  to zero. Furthermore we can translate the affine sphere and apply an affine transformation to obtain  $A(v, w) = \frac{1}{2\sqrt{t_0}}(v, w, f(v, w), 1)$ . After a change of coordinates we get:

$$\varphi(t, v, w) = (tv, tw, tf(v, w) - ct^4, t), \quad c, t > 0. \quad (65)$$

Next we consider the case that  $-a_{22} = a_4 =: a > 0$ . Again we use that  $M^3$  admits a warped product structure and we fix a parameter  $t_0$ . A look at (37)–(41) suggests to define  $\tilde{\xi} = -2aT + \xi$ , then we get at the point  $t_0$ :

$$\begin{aligned} D_V V &= +a_6 V - b_{23} W + \tilde{\xi}, \\ D_V W &= +b_{23} V - a_6 W, \\ D_W V &= -(b_{33} + a_6) W, \\ D_W W &= +(b_{33} - a_6) V + \tilde{\xi}, \\ D_U \tilde{\xi} &= 0. \end{aligned}$$

Thus, if  $v$  and  $w$  are local coordinates which span the second distribution  $L_2$ , then we can interpret  $\varphi(t_0, v, w)$  as a positive definite improper affine sphere in a 3-dimensional linear subspace.

Moreover, we see that this improper affine sphere is a paraboloid provided that  $a_6(t_0, v, w)$  vanishes identically. From the differential equations (43) determining  $a_6$ , we see that this is the case exactly when  $a_6$  vanishes identically, i.e. when  $M^3$  admits a pointwise  $SO(2)$ -symmetry.

After applying an affine transformation and a change of coordinates, we may assume that

$$\varphi(t_0, v, w) = (v, w, f(v, w), 0), \quad (66)$$

with affine normal

$$\tilde{\xi}(t_0, v, w) = (0, 0, 1, 0). \quad (67)$$

We have considered  $\tilde{\xi}$  before. We can solve (48) ( $\frac{\partial}{\partial t}\beta = -2a\beta$ ) explicitly by  $\beta = c\frac{1}{\sqrt{a}}$  (cf. (42)) and get by (49)–(51) that  $D_X(\frac{1}{\sqrt{a}}\tilde{\xi}) = 0$ . Thus  $\frac{1}{\sqrt{a}}(-2aT + \xi) =: C$  for a constant vector  $C$ , i.e.  $T = -\frac{1}{2a}(\sqrt{a}C - \xi)$ . Notice that by (41)  $\xi$  is a constant vector, too. We can choose  $a = \frac{1}{4|t|}$ ,  $t < 0$  (cf. (42)), and we obtain the ordinary differential equation:

$$\frac{\partial}{\partial t}\varphi = -\sqrt{|t|}C - 2t\xi, \quad t < 0. \quad (68)$$

The solution (after a translation) with respect to the initial condition (66) is  $\varphi(t, v, w) = \frac{2}{3}|t|^{\frac{3}{2}}C - t^2\xi + (v, w, f(v, w), 0)$ . Notice that  $C$  is a multiple of  $\tilde{\xi}$  and hence by (67) a constant multiple of  $(0, 0, 1, 0)$ . Furthermore  $\xi$  is transversal to the space spanned by  $\varphi(t_0, v, w)$ . So we get after an affine transformation and a change of coordinates:

$$\varphi(t, v, w) = (v, w, f(v, w) + ct^3, t^4), \quad c, t > 0. \quad (69)$$

Combining both results (65) and (69) we have:

**Theorem 7.** *Let  $M^3$  be an indefinite improper affine hypersphere of  $\mathbb{R}^4$  which admits a pointwise  $\mathbb{Z}_3$ - or  $SO(2)$ -symmetry. Let  $a_{22}^2 = a_4^2$  on  $M^3$ . Then  $M^3$  is affine equivalent with either*

$$\begin{aligned} \varphi : I \times N^2 \rightarrow \mathbb{R}^4 : (t, v, w) &\mapsto (tv, tw, tf(v, w) - ct^4, t), & (a_{22} = a_4) &\quad \text{or} \\ \varphi : I \times N^2 \rightarrow \mathbb{R}^4 : (t, v, w) &\mapsto (v, w, f(v, w) + ct^3, t^4), & (-a_{22} = a_4), & \end{aligned}$$

where  $\psi : N^2 \rightarrow \mathbb{R}^3 : (v, w) \mapsto (v, w, f(v, w))$  is a positive definite improper affine sphere with affine normal  $(0, 0, 1)$  and  $c, t \in \mathbb{R}^+$ . Moreover, if  $M^3$  admits a pointwise  $SO(2)$ -symmetry then  $N^2$  is an elliptic paraboloid.

The computations for the converse statement can be done completely analogous to the previous cases, they even are simpler (the curve is given parametrized).

**Theorem 8.** *Let  $\psi : N^2 \rightarrow \mathbb{R}^3 : (v, w) \mapsto (v, w, f(v, w))$  be a positive definite improper affine sphere with affine normal  $(0, 0, 1)$ . Define  $\varphi(t, v, w) = (tv, tw, tf(v, w) - ct^4, t)$  or  $\varphi(t, v, w) = (v, w, f(v, w) + ct^3, t^4)$ , where  $c, t \in \mathbb{R}^+$ . Then  $\varphi$  defines a 3-dimensional indefinite improper affine hypersphere, which admits a pointwise  $\mathbb{Z}_3$ - or  $SO(2)$ -symmetry.*

## 6 Pointwise $SO(1, 1)$ -symmetry

Let  $M^3$  be a hypersphere admitting a  $SO(1, 1)$ -symmetry. We only state the classification results. The proofs are done quite similar, using a lightvector-frame instead of an orthonormal one, and will appear elsewhere. We denote a lightvector-frame by  $\{E, V, F\}$ , where  $E$  and  $F$  are lightvectors and  $V$  is spacelike (cf. [13]).

**Lemma 13.** *Let  $M^3$  be an affine hypersphere in  $\mathbb{R}^4$  which admits a pointwise  $SO(1, 1)$ -symmetry. Let  $p \in M$ . Then there exists a lightvector-frame  $\{E, V, F\}$  defined in a neighborhood of the point  $p$  and a positive function  $b_4$  such that  $K$  is given by:*

$$\begin{aligned} K(V, V) &= -2b_4V, & K(V, E) &= b_4E, & K(V, F) &= b_4F, \\ K(E, E) &= 0, & K(E, F) &= b_4V, & K(F, F) &= 0. \end{aligned}$$

In the following we denote the coefficients of the Levi-Civita connection with respect to this frame by:

$$\begin{aligned} \widehat{\nabla}_E E &= a_{11}E + b_{11}V, & \widehat{\nabla}_E V &= a_{12}E - b_{11}F, & \widehat{\nabla}_E F &= -a_{12}V - a_{11}F, \\ \widehat{\nabla}_V E &= a_{21}E + b_{21}V, & \widehat{\nabla}_V V &= a_{22}E - b_{21}F, & \widehat{\nabla}_V F &= -a_{22}V - a_{21}F, \\ \widehat{\nabla}_F E &= a_{31}E + b_{31}V, & \widehat{\nabla}_F V &= a_{32}E - b_{31}F, & \widehat{\nabla}_F F &= -a_{32}V - a_{31}F. \end{aligned}$$

Similar as before, it turns out that the vector field  $V$  is globally defined, and we can define the distributions  $L_1 = \text{span}\{V\}$  and  $L_2 = \text{span}\{E, F\}$ . Again,  $L_1$  is autoparallel with respect to  $\widehat{\nabla}$ , and  $L_2$  is spherical with mean curvature normal  $-a_{12}V$ . We introduce a coordinate function  $v$  by  $\frac{\partial}{\partial v} := V$ .

**Lemma 14.** *The function  $\nu = b_4^2 - a_{12}^2 - H$  is globally defined,  $\frac{\partial}{\partial v}(e^{2f}\nu) = 0$  and  $\nu$  vanishes identically or nowhere on  $\mathbb{R}$ .*

Again we have to distinguish three cases.

### 6.1 The first case: $\nu \neq 0$ on $M^3$

**Theorem 9.** Let  $M^3$  be an indefinite affine hypersphere of  $\mathbb{R}^4$  which admits a pointwise  $SO(1, 1)$ -symmetry. Let  $b_4^2 - a_{12}^2 \neq H$  for some  $p \in M^3$ . Then  $M^3$  is affine equivalent to

$$\varphi : I \times N^2 \rightarrow \mathbb{R}^4 : (v, x, y) \mapsto (\gamma_1(v), \gamma_2(v)\phi(x, y)),$$

where  $\phi : N^2 \rightarrow \mathbb{R}^3$  is a one-sheeted hyperboloid and  $\gamma : I \rightarrow \mathbb{R}^2$  is a curve.

**Theorem 10.** Let  $\phi : N^2 \rightarrow \mathbb{R}^3$  be a one-sheeted hyperboloid and let  $\gamma : I \rightarrow \mathbb{R}^2$  be a curve. Define  $\varphi : I \times N^2 \rightarrow \mathbb{R}^4$  by  $\varphi(v, x, y) = (\gamma_1(v), \gamma_2(v)\phi(x, y))$ .

- (i) If  $\gamma = (\gamma_1, \gamma_2)$  satisfies  $\gamma_2^2|\gamma_1\gamma_2' - \gamma_1'\gamma_2|^5 = |\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'|(\gamma_1')^2 \neq 0$ , then  $\varphi$  defines a 3-dimensional indefinite proper affine hypersphere.
- (ii) If  $\gamma = (\gamma_1, \gamma_2)$  satisfies  $\gamma_2^2|\gamma_2'|^5 = |\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'|(\gamma_1')^2 \neq 0$ , then  $\varphi$  defines a 3-dimensional indefinite improper affine hypersphere.

**Theorem 11.** Let  $\phi : N^2 \rightarrow \mathbb{R}^3$  be a one-sheeted hyperboloid and let  $\gamma : I \rightarrow \mathbb{R}^2$  be a curve, such that  $\varphi(v, x, y) = (\gamma_1(v), \gamma_2(v)\phi(x, y))$  defines a 3-dimensional indefinite affine hypersphere. Then  $\varphi(I \times N^2)$  admits a pointwise  $SO(1, 1)$ -symmetry.

### 6.2 The second case: $\nu \equiv 0$ and $H \neq 0$ on $M^3$

**Theorem 12.** Let  $M^3$  be an indefinite proper affine hypersphere of  $\mathbb{R}^4$  which admits a pointwise  $SO(1, 1)$ -symmetry. Let  $H = b_4^2 - a_{12}^2 (\neq 0)$  on  $M^3$ . Then  $M^3$  is affine equivalent with

$$\varphi : I \times N^2 \rightarrow \mathbb{R}^4 : (v, x, y) \mapsto (\gamma_1(v)x, \gamma_1(v)y, \gamma_1(v)f(x, y) + \gamma_2(v), \gamma_1(v)),$$

where  $\psi : N^2 \rightarrow \mathbb{R}^3 : (x, y) \mapsto (x, y, f(x, y))$  is a hyperbolic paraboloid with affine normal  $(0, 0, 1)$  and  $\gamma : I \rightarrow \mathbb{R}^2$  is a curve.

**Theorem 13.** Let  $\psi : N^2 \rightarrow \mathbb{R}^3$  be a hyperbolic paraboloid with affine normal  $(0, 0, 1)$ , and let  $\gamma : I \rightarrow \mathbb{R}^2$  be a curve. Define  $\varphi : I \times N^2 \rightarrow \mathbb{R}^4$  by  $\varphi(v, x, y) = (\gamma_1(v)x, \gamma_1(v)y, \gamma_1(v)f(x, y) + \gamma_2(v), \gamma_1(v))$ . If  $\gamma = (\gamma_1, \gamma_2)$  satisfies  $\gamma_1^2|\gamma_1\gamma_2' - \gamma_1'\gamma_2|^5 = |\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'|(\gamma_1')^2 \neq 0$ , then  $\varphi$  defines a 3-dimensional indefinite proper affine hypersphere.

**Theorem 14.** Let  $\psi : N^2 \rightarrow \mathbb{R}^3$  be a hyperbolic paraboloid with affine normal  $(0, 0, 1)$ , and let  $\gamma : I \rightarrow \mathbb{R}^2$  be a curve, such that  $\varphi(v, x, y) = (\gamma_1(v)x, \gamma_1(v)y, \gamma_1(v)f(x, y) + \gamma_2(v), \gamma_1(v))$  defines a 3-dimensional indefinite proper affine hypersphere. Then  $\varphi(I \times N^2)$  admits a pointwise  $SO(1, 1)$ -symmetry.

### 6.3 The third case: $\nu \equiv 0$ and $H = 0$ on $M^3$

**Theorem 15.** Let  $M^3$  be an indefinite improper affine hypersphere of  $\mathbb{R}^4$  which admits a pointwise  $SO(1, 1)$ -symmetry. Let  $a_{12}^2 = b_4^2$  on  $M^3$ . Then  $M^3$  is affine equivalent with either

$$\begin{aligned} \varphi : I \times N^2 \rightarrow \mathbb{R}^4 : (v, x, y) &\mapsto (vx, vy, vf(x, y) - cv^4, v), & (a_{12} = b_4) &\quad \text{or} \\ \varphi : I \times N^2 \rightarrow \mathbb{R}^4 : (v, x, y) &\mapsto (x, y, f(x, y) + cv^3, v^4), & (-a_{12} = b_4), & \end{aligned}$$

where  $\psi : N^2 \rightarrow \mathbb{R}^3 : (v, w) \mapsto (v, w, f(v, w))$  is a hyperbolic paraboloid with affine normal  $(0, 0, 1)$  and  $c, t \in \mathbb{R}^+$ .

**Theorem 16.** Let  $\psi : N^2 \rightarrow \mathbb{R}^3 : (v, w) \mapsto (v, w, f(v, w))$  be a hyperbolic paraboloid with affine normal  $(0, 0, 1)$ . Define  $\varphi(t, v, w) = (tv, tw, tf(v, w) - ct^4, t)$  or  $\varphi(t, v, w) = (v, w, f(v, w) + ct^3, t^4)$ , where  $t \in \mathbb{R}^+$ ,  $c \neq 0$ . Then  $\varphi$  defines a 3-dimensional indefinite improper affine hypersphere, which admits a pointwise  $SO(1, 1)$ -symmetry.

## A Computations for pointwise $SO(2)$ -, $S_3$ - or $\mathbb{Z}_3$ -symmetry

$$e[1]:=\{1, 0, 0\}; e[2]:=\{0, 1, 0\}; e[3]:=\{0, 0, 1\};$$

$$\text{ONB of } SO(1,2): e[1]=T, e[2]=V, e[3]=W$$

**Affine metric**

$$h[y_-, z_-]:=-y[[1]]z[[1]]+\text{Sum}[y[[i]]z[[i]], \{i, 2, 3\}];$$

**Difference tensor (r=a1, s=a4, u=a6)**

$$\begin{aligned} K[y_-, z_-]:&= \text{Sum}[y[[i]]z[[j]]k[i, j], \{i, 1, 3\}, \{j, 1, 3\}]; \\ k[1, 1]:&=\{-r, 0, 0\}; k[1, 2]:=\{0, s, 0\}; k[1, 3]:=\{0, 0, r-s\}; \\ k[2, 1]:&=\{0, s, 0\}; k[2, 2]:=\{-s, u, 0\}; k[2, 3]:=\{0, 0, -u\}; \\ k[3, 1]:&=\{0, 0, r-s\}; k[3, 2]:=\{0, 0, -u\}; k[3, 3]:=\{-(r-s), -u, 0\}; \end{aligned}$$

**Ricci tensor, scalar curvature and Pick invariant**

$$[Kx, Ky]z$$

$$\begin{aligned} LK[x_-, y_-, z_-]:&=K[x, K[y, z]] - K[y, K[x, z]]; \\ \text{ListLK}:&=\{LK[e[1], e[2], e[1]], LK[e[1], e[2], e[2]], LK[e[1], e[2], e[3]], \\ &LK[e[1], e[3], e[1]], LK[e[1], e[3], e[2]], LK[e[1], e[3], e[3]], \\ &LK[e[2], e[3], e[1]], LK[e[2], e[3], e[2]], LK[e[2], e[3], e[3]]\}; \\ \text{FullSimplify}[\text{ListLK}] \end{aligned}$$

**Curvature tensor (of the Levi-Civita connection)**

$$\begin{aligned} R[x_-, y_-, z_-]:&=H(h[y, z]x - h[x, z]y) - K[x, K[y, z]] + K[y, K[x, z]]; \\ \text{ListR}:&=\{R[e[1], e[2], e[1]], R[e[1], e[2], e[2]], R[e[1], e[2], e[3]], \\ &R[e[1], e[3], e[1]], R[e[1], e[3], e[2]], R[e[1], e[3], e[3]], \\ &R[e[2], e[3], e[1]], R[e[2], e[3], e[2]], R[e[2], e[3], e[3]]\}; \\ \text{Simplify}[\text{ListR}] \end{aligned}$$

**Ricci tensor (of the Levi-Civita connection)**

$$\begin{aligned} \text{ric}[x_-, y_-]:&=\text{Simplify}[(-h[R[e[1], x, y], e[1]] + h[R[e[2], x, y], e[2]] + h[R[e[3], x, y], e[3]]]); \\ \text{Listric}:&=\{\text{ric}[e[1], e[1]], \text{ric}[e[1], e[2]], \text{ric}[e[1], e[3]], \text{ric}[e[2], e[2]], \text{ric}[e[2], e[3]], \text{ric}[e[3], e[3]]\}; \\ \text{Simplify}[\text{Listric}] \end{aligned}$$

**Scalar curvature (of the Levi-Civita connection)**

$$\begin{aligned} \text{sc}:&=1/6(-\text{ric}[e[1], e[1]] + \text{ric}[e[2], e[2]] + \text{ric}[e[3], e[3]]); \\ \text{Simplify}[\text{sc}] \end{aligned}$$

**Pick invariant**

$$\begin{aligned} P:&=1/6(-(k[1, 1][[1]])^2 + (k[2, 2][[2]])^2 + (k[3, 3][[3]])^2 + 3((k[1, 1][[2]])^2 + (k[1, 1][[3]])^2 - \\ &(k[2, 2][[1]])^2 - (k[3, 3][[1]])^2 + (k[2, 2][[3]])^2 + (k[3, 3][[2]])^2) - 6(k[1, 2][[3]])^2); \\ \text{Simplify}[P] \end{aligned}$$

**Lemma 1**

$r = 2s$ ; Simplify[Listric]  
Simplify[P]

**Lemma 8**

$R[e[2], e[3], e[2]]$

**Lemma 3****Levi-Civita connection (ONB)**

$n[1, 1] := \{0, a12, a13\}; n[1, 2] := \{a12, 0, -b13\}; n[1, 3] := \{a13, b13, 0\};$   
 $n[2, 1] := \{0, a22, a23\}; n[2, 2] := \{a22, 0, -b23\}; n[2, 3] := \{a23, b23, 0\};$   
 $n[3, 1] := \{0, a32, a33\}; n[3, 2] := \{a32, 0, -b33\}; n[3, 3] := \{a33, b33, 0\};$   
 $Na[y-, z-] := \text{Sum}[y[[i]]z[[j]]n[i, j], \{i, 1, 3\}, \{j, 1, 3\}] + \text{Sum}[y[[1]]Dt[z[[i]], f1]e[i]$   
 $+ y[[2]]Dt[z[[i]], f2]e[i] + y[[3]]Dt[z[[i]], f3]e[i], \{i, 1, 3\}];$

**Codazzi for K (affine hypersphere)**

$\text{codazzi}[x-, y-, z-] := Na[x, K[y, z]] - K[Na[x, y], z] - K[y, Na[x, z]] - Na[y, K[x, z]] + K[Na[y, x], z] +$   
 $K[x, Na[y, z]];$   
 $\text{eq1} := \text{Simplify}[\text{codazzi}[e[2], e[1], e[1]]]; \text{eq2} := \text{Simplify}[\text{codazzi}[e[3], e[1], e[1]]];$   
 $\text{eq3} := \text{Simplify}[\text{codazzi}[e[1], e[2], e[2]]]; \text{eq4} := \text{Simplify}[\text{codazzi}[e[3], e[2], e[2]]];$   
 $\text{eq5} := \text{Simplify}[\text{codazzi}[e[1], e[3], e[3]]]; \text{eq6} := \text{Simplify}[\text{codazzi}[e[2], e[3], e[3]]];$   
 $\text{eq7} := \text{Simplify}[\text{codazzi}[e[1], e[2], e[3]]]; \text{eq8} := \text{Simplify}[\text{codazzi}[e[2], e[3], e[1]]];$   
 $\text{eq9} := \text{Simplify}[\text{codazzi}[e[3], e[1], e[2]]];$   
 $\text{eq} := \{\text{eq1}, \text{eq2}, \text{eq3}, \text{eq4}, \text{eq5}, \text{eq6}, \text{eq7}, \text{eq8}, \text{eq9}\}; \text{eq}$

**1. case:  $u^2 \neq 4s^2$** **conclusions from eq1,2,4:**

$\text{Simplify}[\text{eq2}[[1]] - 2\text{eq4}[[1]]]$   
 $\text{Simplify}[\text{eq1}[[3]] + \text{eq2}[[2]]]$   
 $a13 = 0; a32 = -a23;$   
 $\text{eq}$

**conclusions from eq1,2,6:**

$\text{Simplify}[-2\text{eq6}[[1]] + \text{eq1}[[1]]]$   
 $\text{Simplify}[\text{eq1}[[2]] - \text{eq2}[[3]]]$   
 $a12 = 0; a33 = a22;$   
 $\text{eq}$   
 $\text{Clear}[a13, a12, a32, a33]$

**2. case:  $u = 2s \neq 0$** 

$u = 2s; \text{eq}$

**conclusions from eq8, eq1:**

$a32 = a23; a13 = -2a23; \text{eq}$

**conclusions from eq3:**

$b13 = 0; \text{eq}$   
 $\text{Simplify}[\text{eq1}[[2]] - \text{eq2}[[3]]]$   
 $a12 = -(a33 - a22); \text{eq}$

Simplify[eq3[[2]] - 1/2eq1[[1]] + 2eq1[[2]]]  
a33 = -a22; eq

It follows that  $T(a_4)=0$ ,  $V(a_4)=-4a_{22} a_4$ ,  $W(a_4)=4a_{23} a_4$ .

Simplify[eq4[[2]] + eq2[[1]]]  
Simplify[eq4[[3]] + eq1[[1]]]  
b23 = -a23; b33 = -a22;

## Lemma 4

### Gauss for Levi-Civita connection (affine hypersphere)

gaussLC[x\_, y\_, z\_] := Na[x, Na[y, z]] - Na[y, Na[x, z]] - Na[Na[x, y] - Na[y, x], z] - Hh[y, z]x + Hh[x, z]y + K[x, K[y, z]] - K[y, K[x, z]];  
eq11 := Simplify[gaussLC[e[1], e[2], e[2]]]; eq12 := Simplify[gaussLC[e[1], e[3], e[2]]];  
eq13 := Simplify[gaussLC[e[2], e[3], e[2]]]; eq14 := Simplify[gaussLC[e[1], e[2], e[1]]];  
eq15 := Simplify[gaussLC[e[1], e[3], e[1]]]; eq16 := Simplify[gaussLC[e[2], e[3], e[1]]];  
eq17 := Simplify[gaussLC[e[1], e[2], e[3]]]; eq18 := Simplify[gaussLC[e[1], e[3], e[3]]];  
eq19 := Simplify[gaussLC[e[2], e[3], e[3]]];  
eqG := {eq11, eq12, eq13, eq14, eq15, eq16, eq17, eq18, eq19};

### 2. case: $u=2s \neq 0$

eqG  
Simplify[eq11[[1]] - eq12[[3]]]  
Simplify[eq15[[3]] + eq12[[3]]]  
Contradiction to eq13.3  
Clear[b33, b23, a33, a12, b13, a32, a13, u]

### 1. case: $u^2 \neq 4s^2$

a13 = 0; a32 = -a23; a12 = 0; a33 = a22; eqG

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